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2001 J. Phys. A: Math. Gen. 34 L441

(http://iopscience.iop.org/0305-4470/34/33/101)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) L441–L446 PII: S0305-4470(01)23396-4

LETTER TO THE EDITOR

On the Baker–Akhiezer function in the AKNS scheme

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Received 23 March 2001, in final form 2 May 2001
Published 10 August 2001
Online at stacks.iop.org/JPhysA/34/L441

Abstract
Expressions for the Baker–Akhiezer function and their logarithmic space and time derivatives are derived in terms of the matrix elements of $\mathbb{U}-\mathbb{V}$ matrices and 'squared basis functions'. These expressions generalize the well known formulas for the KdV equation case and establish links between different forms of the Whitham averaging procedure.

PACS number: 02.30.Jr

It is well known that the Baker–Akhiezer (BA) function plays a central role in the finite-gap integration method of completely integrable equations (see, e.g., [1]). This method permits one to obtain the BA function in terms of Riemann θ-functions and, consequently, a quasi-periodic solution of the equation under consideration.

At the same time, some other expressions for the BA function are of great importance in the general theory of the finite-gap integration method and its applications. As a key example we note the well known formulas from the theory of KdV equation hierarchy

$$u_t = \frac{1}{2}\mathcal{B}_{xxx} + 2\mathcal{B}_x(u+\lambda) + \mathcal{B}u_x \tag{1}$$

which is a compatibility condition of two linear equations

$$\psi_{xx} = -(u+\lambda)\psi \qquad \psi_t = -\frac{1}{2}\mathcal{B}_x\psi + \mathcal{B}\psi_x \tag{2}$$

where $\mathcal{B} = 4\lambda - 2u$ for the KdV equation $u_t + 6uu_x + u_{xxx} = 0$, and which can easily be found for higher KdV equations by means of simple recursion relations (see, e.g., [2–4]). Then the solution of equations (2) can be written in the form

$$\psi^{\pm} = \sqrt{g}\exp\left(\pm \mathrm{i}\sqrt{P(\lambda)}\int^x \frac{\mathrm{d}x}{g}\right) \tag{3}$$

where the product $g = \psi_+\psi_-$ of two basis functions satisfies the equations

$$g_{xxx} + 2u_x g + 4(u+\lambda)g_x = 0 \qquad g_t = \mathcal{B}g_x - \mathcal{B}_x g. \tag{4}$$

The first equation here has the first integral

$$\tfrac{1}{2}gg_{xx} - \tfrac{1}{4}g_x^2 + (u+\lambda)g^2 = P(\lambda) \tag{5}$$

where $P(\lambda)$ is an odd-degree polynomial in λ. If $P(\lambda)$ is the $(2n+1)$-degree polynomial

$$P(\lambda) = \prod_{i=1}^{2n+1}(\lambda - \lambda_i) = \lambda^{2n+1} - s_1\lambda^{2n} + s_2\lambda^{2n-1} + \cdots + s_{2n}\lambda - s_{2n+1} \tag{6}$$

then g is the n-degree polynomial

$$g = \prod_{i=1}^{n}(\lambda - \mu_i) = \lambda^n - \sigma_1\lambda^{n-1} + \sigma_2\lambda^{n-2} + \cdots + (-1)^n\sigma_n \tag{7}$$

whose coefficients are expressed in terms of u and its x derivatives by the trace formulas

$$\begin{aligned} &\sigma_1 = \textstyle\sum \mu_i = \frac{1}{2}(u + s_1) \\ &\sigma_2 = \textstyle\sum_{i<j} \mu_i\mu_j = \frac{1}{4}(u_{xx} + 3u^2) + \frac{3}{2}s_1 u + s_2 - \frac{1}{4}s_1^2 \\ &\ldots . \end{aligned} \tag{8}$$

Substitution of equations (7) and (8) in equation (3) yields the BA function in terms of the potential $u(x,t)$ and its x derivatives. Differentiation of equation (3) with respect to x gives

$$\frac{\psi_x^\pm}{\psi^\pm} \equiv (\ln\psi^\pm)_x = \frac{g_x \pm 2\mathrm{i}\sqrt{P(\lambda)}}{2g}. \tag{9}$$

From the second equation (4) we have the conservation law

$$\left(\frac{1}{g}\right)_t = \left(\frac{\mathcal{B}}{g}\right)_x \tag{10}$$

and, hence, differentiation of equation (3) with respect to t gives

$$\frac{\psi_t^\pm}{\psi^\pm} \equiv (\ln\psi^\pm)_t = \frac{g_t \pm 2\mathrm{i}\sqrt{P(\lambda)}\,\mathcal{B}}{2g}. \tag{11}$$

Thus, the logarithmic derivatives of the BA fuctions provide the generating functions of densities and flows for sequence of conservation laws of the KdV equation[1].

A natural question arises: how one can generalize the formulas (3), (9) and (11) for the BA function and its logarithmic derivatives on other integrable systems. Some particular results in this direction were obtained in [6] for the AKNS hierarchies corresponding to the Zakharov–Shabat spectral problem. Here we shall derive by a simple and direct method the formulas of that kind for the general form of the 2×2 AKNS scheme without any specializing of the '$\mathbb{U} - \mathbb{V}$' matrices.

We shall start from two linear systems

$$\begin{aligned} \psi_{1,x} &= F\psi_1 + G\psi_2 \qquad & \psi_{1,t} &= A\psi_1 + B\psi_2 \\ \psi_{2,x} &= H\psi_1 - F\psi_2 \qquad & \psi_{2,t} &= C\psi_1 - A\psi_2 \end{aligned} \tag{12}$$

which constitute the AKNS scheme (in [7] this was discussed for a particular case with $F = -\mathrm{i}\lambda$, $G = q(x,t)$, $H = r(x,t)$), where the coefficients depend on an arbitrary spectral

[1] Note that the equations (3), (5) and (9), as well as some other facts from the finite-gap integration method, were discovered as early as in 1919 by J Drach in his remarkable but forgotten papers [5]. I am grateful to Yu V Brezhnev for information about these papers.

parameter λ and functions $u_k(x,t)$ whose evolution is governed by the equations resulting from the compatibility conditions $\psi_{1,xt} = \psi_{1,tx}$, namely,

$$\begin{aligned} &F_t - A_x + CG - BH = 0 \\ &G_t - B_x + 2(BF - AG) = 0 \\ &H_t - C_x + 2(AH - CF) = 0. \end{aligned} \tag{13}$$

The systems (12) have two basis solutions $\psi^{\pm} = (\psi_1^{\pm}, \psi_2^{\pm})$ from which 'squared basis functions'

$$f = -\frac{\mathrm{i}}{2}\left(\psi_1^+\psi_2^- + \psi_1^-\psi_2^+\right) \qquad g = \psi_1^+\psi_1^- \qquad h = -\psi_2^+\psi_2^- \tag{14}$$

can be constructed. They satisfy the following linear systems:

$$\begin{aligned} &f_x = -\mathrm{i}Hg + \mathrm{i}Gh && f_t = -\mathrm{i}Cg + \mathrm{i}Bh \\ &g_x = 2\mathrm{i}Gf + 2Fg && g_t = 2\mathrm{i}Bf + 2Ag \\ &h_x = -2\mathrm{i}Hf - 2Fh && h_t = -2\mathrm{i}Cf - 2Ah. \end{aligned} \tag{15}$$

From equations (13) and (15) the important relations

$$\begin{aligned} &\left(\frac{G}{g}\right)_t = \left(\frac{B}{g}\right)_x \\ &\left(\frac{H}{h}\right)_t = \left(\frac{C}{h}\right)_x \end{aligned} \tag{16}$$

can be obtained [8] which represent the generating functions for conservation laws of the evolution equations (13). The constancy of the Wronskian of systems (12) yields the relation

$$-\tfrac{1}{4}(\psi_1^+\psi_2^- - \psi_1^-\psi_2^+)^2 = f^2 - gh = P(\lambda) \tag{17}$$

and periodic solutions of equations (13) are distinguished by the condition that $P(\lambda)$ be a polynomial in λ.

Our aim is to obtain formulas analogous to equations (3), (9) and (11) for $\psi^{\pm}$. To this end we notice that the square root from equation (17),

$$\frac{\mathrm{i}}{2}(\psi_1^+\psi_2^- - \psi_1^-\psi_2^+) = \sqrt{P(\lambda)}$$

(another choice of sign interchanges ψ^+ and ψ^-) and the first equation of (14) yield the identities

$$\frac{-\mathrm{i}\psi_1^+\psi_2^-}{f - \sqrt{P(\lambda)}} = \frac{-\mathrm{i}\psi_1^-\psi_2^+}{f + \sqrt{P(\lambda)}} = 1 \tag{18}$$

and that equation (17) can be rewritten in the form

$$gh = \big(f - \sqrt{P(\lambda)}\big)\big(f + \sqrt{P(\lambda)}\big). \tag{19}$$

Then we have a chain of simple transformations

$$\begin{aligned} \psi_{1,x}^+ &= F\psi_1^+ + G\psi_2^+ \\ &= \frac{1}{2g}(g_x - 2\mathrm{i}Gf)\psi_1^+ + G\frac{-\mathrm{i}\psi_1^+\psi_2^-}{f - \sqrt{P(\lambda)}}\psi_2^+ && \text{by (15) and (18)} \\ &= \frac{1}{2g}\left[g_x - 2\mathrm{i}G\left(f - \frac{gh}{f - \sqrt{P(\lambda)}}\right)\right]\psi_1^+ && \text{by (14)} \\ &= \frac{1}{2g}(g_x + 2\mathrm{i}\sqrt{P(\lambda)}\,G)\psi_1^+ && \text{by (19)} \end{aligned}$$

and similar calculations can be done for the other x derivatives of $\psi^{\pm}$. As a result we obtain the formulas

$$\begin{aligned}\psi^{\pm}_{1,x} &= \frac{1}{2g}\big(g_x \pm 2\mathrm{i}\sqrt{P(\lambda)}\,G\big)\psi^{\pm}_1\\ \psi^{\pm}_{2,x} &= \frac{1}{2h}\big(h_x \pm 2\mathrm{i}\sqrt{P(\lambda)}\,H\big)\psi^{\pm}_1\end{aligned}\tag{20}$$

for which integration yields

$$\psi^{\pm}_1 = \sqrt{g}\,\exp\left(\pm\,\mathrm{i}\sqrt{P(\lambda)}\int^x \frac{G}{g}\mathrm{d}x\right)\tag{21}$$

$$\psi^{\pm}_2 = \sqrt{-h}\,\exp\left(\pm\,\mathrm{i}\sqrt{P(\lambda)}\int^x \frac{H}{h}\mathrm{d}x\right).\tag{22}$$

Differentiation of these relations with respect to t gives, taking into account equations (16),

$$\begin{aligned}\psi^{\pm}_{1,t} &= \frac{1}{2g}\big(g_t \pm 2\mathrm{i}\sqrt{P(\lambda)}\,B\big)\psi^{\pm}_1\\ \psi^{\pm}_{2,t} &= \frac{1}{2h}\big(h_t \pm 2\mathrm{i}\sqrt{P(\lambda)}\,C\big)\psi^{\pm}_2.\end{aligned}\tag{23}$$

These are the formulas for the BA functions we wanted to obtain.

As a simple application of these results let us note that equations (20) and (23) written in the form

$$\begin{aligned}\left(\ln\frac{\psi^{+}_1}{\sqrt{g}}\right)_x &= \pm\,\mathrm{i}\sqrt{P(\lambda)}\,\frac{G}{g}\\ \left(\ln\frac{\psi^{+}_1}{\sqrt{g}}\right)_t &= \pm\,\mathrm{i}\sqrt{P(\lambda)}\,\frac{B}{g}\end{aligned}\tag{24}$$

(and similar formulas for $\psi^{\pm}_2$) show that in the AKNS scheme the Whitham averaging over fast variation of the quasi-periodic solution with slowly modulated parameters (zeros of the polynomial $P(\lambda)$) performed according to Krichever's rule (see, e.g., [9])

$$\left[\overline{\left(\ln\frac{\psi^{+}_1}{\sqrt{g}}\right)_x}\right]_t = \left[\overline{\left(\ln\frac{\psi^{+}_1}{\sqrt{g}}\right)_t}\right]_x\tag{25}$$

is equivalent to the rule suggested in [8] (see also [4])

$$\left(\overline{\sqrt{P(\lambda)}\,\frac{G}{g}}\right)_t = \left(\overline{\sqrt{P(\lambda)}\,\frac{B}{g}}\right)_x\tag{26}$$

where the line over an expression denotes its average.

Let us illustrate these results by a simple concrete example of one-phase periodic solution of the nonlinear Schrödinger (NLS) equation

$$\mathrm{i}u_t + u_{xx} + 2|u|^2u = 0\tag{27}$$

which can be presented as a compatibility condition of the linear systems (12) with the coefficients

$$\begin{aligned}&F = -\mathrm{i}\lambda \qquad G = \mathrm{i}u \qquad H = \mathrm{i}u^*\\ &A = -2\mathrm{i}\lambda^2 + \mathrm{i}|u|^2 \qquad B = 2\mathrm{i}u\lambda - u_x \qquad C = 2\mathrm{i}u^*\lambda + u^*_x.\end{aligned}\tag{28}$$

The one-phase solution corresponds to the fourth-degree polynomial (17)

$$P(\lambda) = \lambda^4 - s_1\lambda^3 + s_2\lambda^2 - s_3\lambda + s_4\tag{29}$$

(see, e.g., [4]), and in this case the solution of the system (15) is given by

$$f = \lambda^2 - f_1\lambda + f_2 \qquad g = \mathrm{i}u(\lambda - \mu) \qquad h = \mathrm{i}u^*(\lambda - \mu^*) \tag{30}$$

where

$$f_1 = s_1/2 \qquad f_2 = \left(s_2 - s_1^2/4 - |u|^2\right)/2 \tag{31}$$

and the auxiliary spectrum point μ satisfies the Dubrovin equations

$$\mu_x = -2\mathrm{i}\sqrt{P(\mu)} \qquad \mu_t = s_1\mu_x \tag{32}$$

and is connected with u by the trace formula

$$u_x = 2\mathrm{i}u(\mu - s_1/2). \tag{33}$$

With the use of these relations, the first formula (21) transforms at once to

$$\begin{aligned}\psi_1^{\pm} &= \sqrt{\mathrm{i}u(\lambda - \mu)}\exp\left(\pm\,\mathrm{i}\sqrt{P(\lambda)}\int^x \frac{\mathrm{d}x}{\lambda - \mu}\right)\\ &= \sqrt{\mathrm{i}u(\lambda - s_1/2) - u_x/2}\,\exp\left(\pm\,\mathrm{i}\sqrt{P(\lambda)}\int^x \frac{\mathrm{d}x}{\lambda - s_1/2 + \mathrm{i}(\ln u)_x/2}\right)\end{aligned} \tag{34}$$

and similar expressions can be written for $\psi_2^{\pm}$.

The last line of (34) gives $\psi_1^{\pm}$ in terms of $u,\ u_x$, constant parameters $s_i,\ i = 1, 2, 3, 4$, and the spectral parameter λ. However, in some applications it is more convenient to use the first line of (34) due to simplicity of the corresponding logarithmic derivatives (24). In particular, for the NLS equation case, the averaged generating function (26) of the conservation laws takes the form

$$\left(\overline{\sqrt{P(\lambda)}\,\frac{1}{\lambda - \mu}}\right)_t = \left(\overline{\sqrt{P(\lambda)}\left(2 + \frac{s_1}{\lambda - \mu}\right)}\right)_x \tag{35}$$

and averaging with the use of (32) according to the rule

$$\overline{\frac{1}{\lambda - \mu}} = \frac{1}{L}\oint\frac{\mathrm{d}x}{\lambda - \mu} = \frac{\mathrm{i}}{2L}\oint\frac{\mathrm{d}\mu}{(\lambda - \mu)\sqrt{P(\lambda)}}$$

where L is the wavelength, leads very easily to the Whitham modulational equations (see, e.g., [4]). A similar approach applies to many other equations described by the AKNS inverse scattering scheme.

Finally, I note that in the recent paper [10] a method was suggested for obtaining expressions for BA functions in terms of potentials, which was based on Novikov's representation of the finite-gap solution as a stationary solution of the so-called Lax–Novikov equations involving higher equations of the hierarchy under consideration. A simple method presented here gives the same results after expressing g in terms of potentials $u_k(x, t)$ that can be easily achieved by the use of trace formulas. It does not use higher equations of the hierarchy under consideration, yields also the temporal counterparts of formulas for logarithmic derivatives of the BA functions, but is restricted to the AKNS scheme only.

I am grateful to Yu V Brezhnev for useful discussions. This work was partially supported by RFBR (grant 01–01–00696) and FAPESP (Brazil).

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